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Hopf Modules and Their Duals[1]

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Free Hopf modules and bimodules over a bialgebra are studied. We investigate a duality in the category of bimodules in this context. This gives the correspondence between Woronowicz's quantum Lie algebra and algebraic vector fields.

1. INTRODUCTION

We are interested in comparing the Woronowicz (1989) quantum Lie algebra for a quantum group with the algebraic vector fields for the first-order differential calculus over a unital associative algebra (Borowiec, 1996, 1997). Dualizing a bicovariant bimodule of one-forms in the category of bimodules over a Hopf algebra with bijective antipode, one obtains a bicovariant bimodule of algebraic vector fields, called a Cartan pair (Borowiec, 1996). As in the Lie algebra case, the Woronowicz quantum Lie algebra consists of the left or right invariant vector fields. Since the work of Woronowics (1989), bicovariant differential calculi have become the subject many investigations (Bernard, 1990; Durdevich, 1996, 1997, 2000, Klimyk and Schmüdgen, 1997; Majid, 1998; Oziewicz, 1998). Construction of vector fields for bicovariant differential calculi on Hopf algebras has also been discussed (Aschieri and Schupp,1996; Pflaum and Schauenburg, 1996; Schauenburg, 1996).

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The problem of the generalization of the Lie module to quantum and braided categories is still open. Among several propositions we mention Pareigis' (1996) approach to Lie algebras in the category of Yetter–Drinfeld modules and several approaches based on variety of braided identities generalizing the Jacobi identity(Oziewicz *et al.*, 1994; Nishimura, 1999; Bautista, 2000).

In the sequel, $\Bbbk$ is a field. We shall work in the category of $\Bbbk$-spaces, all maps are $\Bbbk$ linear maps, and the tensor product is over $\Bbbk$. Given $\Bbbk$-spaces U and W, Hom (U, W) denotes the $\Bbbk$-space of all $\Bbbk$-linear maps from U to W. Denote by $\tilde{U} \doteq \mathrm{Hom}\,(U, \Bbbk)$ the $\Bbbk$-linear dual of U. For $\Phi \in \mathrm{Hom}\,(U, W)$, we denote by $\tilde{\Phi} \in \mathrm{Hom}(\tilde{W}, \tilde{U})$ its linear transpose. Dealing with finite-dimensional $\Bbbk$-spaces, we shall use the covariant index notation together with the Einstein summation convention over repeated up contravariant and down covariant indices. If $\{e_k\}_{k=1}^{\dim V}$ denotes a basis in a finite-dimensional $\Bbbk$-space V, then $v = u^i e_i \in V$.

An algebra, means an associative unital $\Bbbk$-algebra and a coalgebra means a coassociative counital $\Bbbk$-coalgebra. If $A \equiv (A, m, 1)$ is an algebra, then A^{op} denotes an algebra with the opposite multiplication: $a \cdot_{\mathrm{op}} b = ba$ Let $C \equiv (C, \Delta, \epsilon)$be a coalgebra with comultiplication Δ and counit ϵ. The Sweedler (1969) notation is $\Delta(a) = a_{(1)} \otimes a_{(2)}$. For a left (right) comultiplication in V, we shall write $\Delta_V(v) = v_{(-1)} \otimes v_{(0)}$ $({}_V\Delta(v) = v_{(0)} \otimes v_{(1)})$. By C^{cop} we mean an opposite coalgebra structure with $\Delta^{\mathrm{cop}}(a) = a_{(2)} \otimes a_{(1)}$. For a bialgebra B, one can form new bialgebras by taking the opposite of either the algebra or/and coalgebra structure, e.g., $B^{\mathrm{op,cop}}$ has both opposite structures.

2. PRELIMINARIES

Let A be an algebra. Assume that a finite-dimensional $\Bbbk$-space V is a left A-module, or equivalently, it is a carrier space for representation λ of A. This means that the left action $m_V: A \otimes V \to V$ can be written in terms of a unital $\Bbbk$-algebra homomorphism $\lambda: A \to \mathrm{End}\, V$, which, in turn, by the use of a basis $\{e_k\}_{k=1}^{\dim V}$ of V, can be rewritten in matrix form $\lambda_k^i(a) e_i \doteq m_V(a \otimes e_k) \doteq \lambda(a) e_k$,

$$\forall a, b \in A, \qquad \lambda_k^i(1) = \delta_k^i, \qquad \lambda_k^i(ab) = \lambda_m^i(a)\lambda_k^m(b) \tag{1}$$

The same matrix representation uniquely induces the transpose right multiplication ${}_{\tilde{V}}m \doteq \widetilde{m_V}: \tilde{V} \otimes A \to \tilde{V}$ on the dual vector space $\tilde{V}$:

$$\widetilde{m_V}(e^k \otimes a) \doteq \tilde{\lambda}(a) e^k \equiv \lambda_m^k(a) e^m \tag{2}$$

where the e^k are elements of the dual basis in $\tilde{V}$. This defines an antirepresentation $\tilde{\lambda}: A \to \mathrm{End}\, \tilde{V}$ given by the transpose matrices $\tilde{\lambda}(a)$: $\tilde{\lambda}(ab) = \tilde{\lambda}(b)\tilde{\lambda}(a)$.

Alternatively, one can say that $\tilde{\lambda}$: $A^{\mathrm{op}} \to \mathrm{End}\, \tilde{V}$ is a representation of the opposite algebra A^{op}, i.e., it defines a left A^{op}-module structure on $\tilde{V}$.

For an algebra morphism T: $A' \to A$ and a left A'-module action m'_L: $A' \otimes V \to V$, one defines its *pullback* as a left A-module action $m_L \doteq T^*(m'_L)$: $A \otimes V \to V$ by

$$T^*(m'_L) \doteq m'_L \circ (T \otimes \mathrm{id}), \qquad a \cdot_T v \doteq T(a) \cdot v \quad \text{or} \quad \lambda_i^k \doteq \lambda_i'^k \circ T \quad (3)$$

When T is an antihomomorphism (i.e., a homomorphism from A into A'^{op}), then the pull/back $T^*(m'_L)$ is a right A-module action.

Let C be a coalgebra. A left C-comodule structure on V is determined by *matrix-like* elements

$$L_k^i \in C, \quad i, k = 1, \ldots, \dim V, \qquad \Delta(L_k^i) = L_k^m \otimes L_m^i, \qquad \epsilon(L_k^i) = \delta_k^i \quad (4)$$

These define the left coaction or corepresentation

$$\Delta_V: V \to C \otimes V, \qquad \Delta_V(e_k) = (e_k)_{(-1)} \otimes (e_k)_{(0)} \doteq L_k^m \otimes e_m \quad (5)$$

The same matrix elements $L_k^i \in C$ induce the transpose right comultiplication ${}_V\Delta \doteq \widetilde{\Delta_V}$: $\tilde{V} \to \tilde{V} \otimes C$,

$$\widetilde{\Delta_V}(e^k) = (e^k)_{(0)} \otimes (e^k)_{(1)} \doteq e^m \otimes L_m^k \quad (6)$$

on the dual vector space $\tilde{V}$ (Borowiec and Vázquez-Coutiño, 2000). Alternatively, one can say that $\widetilde{\Delta_V}$ defines a left coaction of the coopposite coalgebra C^{cop} on $\tilde{V}$.

For a coalgebra morphism T: $C \to C'$ and a left C-comodule coaction Δ_L: $V \to C \otimes V$, one defines its *pushforward* $\Delta'_L \equiv T_*(\Delta_L)$: $V \to C' \otimes V$ as a left C'-comodule coaction such that

$$T_*(\Delta_L) \doteq (T \otimes \mathrm{id}) \circ \Delta_L \qquad \text{or} \qquad L_i'^k = T(L_i^k) \quad (7)$$

If T is an anticoalgebra map, then its pushforward $T_*(\Delta_L)$ is a right C'-coaction on V.

3. YETTER–DRINFELD MODULES

Our basic references on Yetter-Drinfeld ($\mathscr{YD}$) modules are Montgomery (1993), Radford and Towber (1993), and Schauenburg (1994). $\mathscr{YD}$ modules are known also under the name of Yang-Baxter or crossed modules. Various modules and comodules over a bialgebra are our objects of investigation. The most substantial results are obtained for the case of bialgebras with the bijective antipode (quantum groups).

Definition 3.1 (Yetter–Drinfeld module). For bialgebra B, a left-left *Yetter–Drinfeld* module is a $\mathbb{k}$-space $V \equiv (V, m_V, \triangle_V)$ which is both a left B-module and a left B-comodule and satisfies the compatibility condition

$$\forall a \in B \text{ and } v \in V, \qquad a_{(1)}v_{(-1)} \otimes a_{(2)}v_{(2)} = (a_{(1)} \cdot v)_{(-1)}a_{(2)} \otimes (a_{(1)} \cdot v)_{(0)}$$
$$\text{or} \quad a_{(1)}L_k^m\lambda_m^i(a_{(2)}) = L_m^i a_{(2)}\lambda_k^m(a_{(1)}) \tag{8}$$

We denote by ${}^B_B\mathscr{YD}$ the category of left–left $\mathscr{YD}$ modules over B. Similarly, the right–right $\mathscr{YD}$ module condition is

$$v_{(0)}a_{(1)} \otimes v_{(1)}a_{(2)} = (v \cdot a_{(2)})_{(0)} \otimes a_{(1)}(v \cdot (a_{(2)}))_{(1)}$$
$$\text{or} \quad a_{(1)}R_m^k\rho_i^m(a_{(2)}) = R_i^m a_{(2)}\rho_m^k(a_{(1)}) \tag{9}$$

where ${}_vm(e_k \otimes a) \doteq \rho_k^m(a)e_m$ and ${}_v\triangle(e_k) \doteq e_m \otimes R_k^m$ denotes the right multiplication and comultiplication in V. In addition, $\mathscr{YD}^B_B$ denotes the category of right–right $\mathscr{YD}$ B-modules, the category of left–right Yetter-Drinfeld B-modules is denoted by ${}_B\mathscr{YD}^B$, and the category of right–left $\mathscr{YD}$ B-modules by ${}^B\mathscr{YD}_B$.

Remark 3.2. The following categories can be identified in the formal sense:

$${}^B_B\mathscr{YD} \equiv {}_{B^{\mathrm{cop}}}\mathscr{YD}^{B^{\mathrm{cop}}} \equiv {}^{B^{\mathrm{cop}}}\mathscr{YD}_{B^{\mathrm{op}}} \equiv \mathscr{YD}^{B^{\mathrm{op,cop}}}_{B^{\mathrm{op,cop}}} \tag{10}$$

i.e., if a triple $(V, m_L, \triangle_L)$ is an element of the first category, then it becomes automatically (after suitable reinterpretations) an element of the remaining categories. For example, denoting by $\triangle_L^{\mathrm{cop}}(e_k) = e_m \otimes L_k^m$ a canonical right B^{cop} comodule structure on V associated with $\triangle_L$, one sees that $(V, m_L, \triangle_L^{\mathrm{cop}}) \in {}_{B^{\mathrm{cop}}}\mathscr{YD}^{B^{\mathrm{cop}}}$.

Nontrivial category equivalences have been found for the special case when B is a Hopf algebra with bijective antipode.

Proposition 3.3 (Radford and Towber, 1993, p. 265). Suppose that B is a bialgebra with bijective antipode S. Then:

(i) (Woronowicz, 1989) $(V, m_L, \triangle_L) \mapsto (V, (S^{-1})^*(m_L), S_*(\triangle_L))$ describes categorical isomorphisms

$${}^B_B\mathscr{YD} \cong \mathscr{YD}^B_B \qquad \text{and} \qquad {}_B\mathscr{YD}^B \cong {}^B\mathscr{YD}_B \tag{11}$$

(ii) $(V, m_L, \triangle_R) \mapsto (V, m_L, S_*(\triangle_R))$ describes categorical isomorphisms

$${}_B\mathscr{YD}^B \cong {}^B_B\mathscr{YD} \tag{12}$$

Corollary 3.4. Combining (10) and (12), one gets the category equivalences

$$
\begin{aligned}
{}^{B}_{B}\mathcal{YD} \ni (V, m_L, \triangle_L) &\mapsto (V, m_L, (S^{-1})_*(\triangle_L^{\text{cop}})) \in {}^{B^{\text{cop}}}_{B^{\text{cop}}}\mathcal{YD} \\
\mathcal{YD}^{B}_{B} \ni (V, m_R, \triangle_R) &\mapsto (V, m_R, (S^{-1})_*(\triangle_R^{\text{cop}})) \in \mathcal{YD}^{B^{\text{cop}}}_{B^{\text{cop}}} \qquad (13)
\end{aligned}
$$

These follow from the fact that S^{-1} is an antipode of B^{cop}.

Proposition 3.5. Let $(V, m_L, \triangle_L)$ be a left module and left comodule over a bialgebra B, with $\dim V < \infty$. Then $(V, m_L, \triangle_L) \in {}^{B}_{B}\mathcal{YD}$ if and only if $(\tilde{V}, \widetilde{m_L}\widetilde{\triangle_L}) \in \mathcal{YD}^{B}_{B}$.

Proof. Substituting $L^i_k = R^k_i$ and $\lambda^i_k = \rho^k_i$ into equation (8), one gets (9). A right-handed version of the proposition also holds. ∎

Remark 3.6. A $\mathcal{YD}$ structure on V generates the quantum Yang–Baxter operator $\mathcal{R} \in \text{End}\,(V \otimes V)$. For example, if $(V, {}_V m, {}_V\triangle) \in \mathcal{YD}^{B}_{B}$, one gets

$$\mathcal{R}(e_i \otimes e_k) = \rho^j_i(R^m_k)e_m \otimes e_j$$

4. FREE COVARIANT BIMODULES

In the sequel, V is a finite-dimensional $\mathbb{k}$-space spanned by free generators $\xi_1, \ldots, \xi_n$, $n = \dim V$.

We consider left (right) free A-module M represented as $A \otimes V$ ($V \otimes A$), where the module structure is realized by the left (right) multiplication in A. Following Sweedler (1969), we shall use the notation $V_A \doteq V \otimes A$ and ${}_AV \doteq A \otimes V$ for the right and the left free A-modules generated by a vector space V. We do not assume an *invariant basis property* for algebra A. This means that the number of free generators is not necessarily a characteristic number for a given free (left or right) module. In other words, one can have a left A-module isomorphism $A \otimes V \cong A \otimes W$ with $\dim V \neq \dim W$.

The unit $1 \equiv 1_A$ of A enables us to define a canonical inclusion

$$V \ni v \cong v \otimes 1 \in V_A \qquad \text{and} \qquad \forall\, x \in V_A, \quad x = e_i \otimes x^i \qquad (14)$$

where components $x^i \in A$ are uniquely determined with respect to a given basis $\{e_i\}$. Any basis $\{e_i\}$ in V determines a set of free generators $\{\xi_i = e_i \otimes 1_A\}$ in the module V_A.

Lemma 4.1 (Schauenburg, 1994; Borowiec *et al.*, 1997). Let V be a finite-dimensional $\mathbb{k}$-space. The following are equivalent:

(i) A left A-module structure on a free right module V_A such that it becomes an A-bimodule.

(ii) A unital $\mathbb{k}$-algebra map (so-called commutation rule) $\Lambda: A \to A \otimes \text{End}\, V$.

(iii) A $\mathbb{k}$-linear map (so-called twist) $\hat{\Lambda}: {}_AV \to V_A$ such that

$$\hat{\Lambda}(1 \otimes v) = v \otimes 1,$$
$$\hat{\Lambda} \circ (m \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes m) \circ (\hat{\Lambda} \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \hat{\Lambda})$$

Proof. By uniqueness of the decomposition (14), one can set in an arbitrary basis $\{e_i\}$ of V,

$$\Lambda(a)e_k \doteq a \cdot (e_k \otimes 1) \doteq \hat{\Lambda}(a \otimes e_k) \doteq e_i \otimes \Lambda^i_k(a) \tag{15}$$

Properties of the $\mathbb{k}$-algebra map

$$\Lambda^i_k(1) = \delta^i_k, \qquad \Lambda^i_k(ab) = \Lambda^i_m(a)\Lambda^m_k(b) \tag{16}$$

as well as (iii) are to be verified. ■

A right-handed version of the lemma above: A right commutation rule $\Phi: A \to A \otimes \mathrm{End}\, V$ gives rise to a right multiplication on ${}_AV$,

$$(1 \otimes e_k) \cdot a \doteq \Phi^i_k(a) \otimes e_i, \qquad \Phi^i_k(ab) = \Phi^m_k(a)\Phi^i_m(b), \qquad \Phi^i_k(1) = \delta^i_k \tag{17}$$

This implies that Φ is an algebra map from A^{op} into $A^{\mathrm{op}} \otimes \mathrm{End}\, V$.

If C is a coalgebra, then ${}_CV$ is a left free comodule with a comultiplication determined by that in C, i.e., $\Delta_{{}_CV} = \Delta \otimes id_V$. The counit ϵ in C enables us to define a projection map $\epsilon_V: {}_CV \to V$ by $\epsilon_V(x^i \otimes e_i) \doteq \epsilon(x^i)e_i$,

$$\epsilon_V(a \cdot x) = \epsilon(a)\epsilon_V(x), \qquad (\mathrm{id} \otimes \epsilon_V) \circ \Delta_{{}_BV} = \mathrm{id} \tag{18}$$

Let B be a bialgebra. In this case ${}_BV$ is a (free) left module and a (free) left comodule with multiplication and comultiplication satisfying the following compatibility condition:

$$\forall a \in B \text{ and } x \in {}_BV, \qquad \Delta_{{}_BV}(a \cdot x) = \Delta(a)\Delta_{{}_BV}(x) \tag{19}$$
$$(a \cdot x)_{(-1)} \otimes (a \cdot x)_{(0)} = a_{(1)} \cdot x_{(-1)} \otimes a_{(2)}x_{(0)}$$

This condition differs from the $\mathscr{YD}$ conditions and defines a left *Hopf* B-module structure on a left free module ${}_BV$. Similarly, V_B becomes automatically a right free Hopf B-module.

Remark 4.2. B and B^{cop} have the same algebra structure. Therefore we can treat ${}_BV$ as a left free Hopf B^{cop}-module with coaction $\Delta^{\mathrm{cop}}_{{}_BV} = \Delta^{\mathrm{cop}} \otimes id_V$. A $\mathbb{k}$-space V generates a free left (right) either B- or B^{cop}-Hopf module structure on ${}_BV$ (V_B).

Remark 4.3 (Sweedler, 1969). In the case of a Hopf algebra H, any left (right) Hopf H-module is left (right) free, i.e., it has the form ${}_HV$ (V_H), with

V being (not necessarily a finite dimensional) vector space of left (right) invariant elements.

A left Hopf B-module which is at the same time a B-bimodule satisfying

$$\forall x \in {}_BV \quad \text{and} \quad a \in B, \qquad \Delta_{{}_BV}(x \cdot a) = \Delta_{{}_BV}(x)\Delta(a) \tag{20}$$

is called a left *covariant bimodule* (Woronowicz, 1989). The right B-module structure on ${}_BV$ can be used to generate, via the projection map (18), a right B-module structure on the vector space V:

$$\rho(a)v \doteq \epsilon_V((1 \otimes v) \cdot a) \tag{21}$$

Due to (18) and (20), one obtains the following relationship between right module structures on V and on ${}_BV$:

$$(a \otimes v) \cdot b = ab_{(1)} \otimes \rho(b_{(2)})v \tag{22}$$

This means that the converse statement is also true: any right B-module structure ρ on V generates a left B-covariant bimodule structure on a left free Hopf B-module ${}_BV$. Similarly, the formula

$$(a \otimes v) \cdot b = ab_{(2)} \otimes \rho(b_{(1)})v \tag{23}$$

induces a left covariant B^{cop}-bimodule structure on V_B. In other words, there is a bijective correspondence between right module structures on V and left covariant either B-or B^{cop}-bimodule structures on ${}_BV$. For a free right B-covariant bimodule V_B, one gets instead

$$a \cdot (v \otimes b) = \lambda(a_{(1)})v \otimes a_{(2)}b \tag{24}$$

where λ denotes the left B-module structure induced on V.

The following version of Lemma 4.1 is essentially due to Woronowicz (1989).

Proposition 4.4 (Woronowicz 1989). Let V be a finite-dimensional $\mathbb{k}$-space and B be a $\mathbb{k}$-bialgebra. Then the following are equivalent:

(i) A left B-module structure λ: $B \to \text{End } V$.

(ii) A left B-module structure on a right free Hopf B-module V_B such that it becomes a right (free) B-covariant bimodule. Moreover, the commutation rule (16) takes the form $\Lambda^i_k(a) = \lambda^i_k(a_{(1)})a_{(2)}$. Conversely, $\lambda^i_k = \epsilon \circ \Lambda^i_k$.
(iii) A left B $(=B^{\text{cop}})$-module structure on a right free Hopf B^{cop}-module V_B such that it becomes a right (free) B^{cop}-covariant bimodule. In this case, the commutation rule takes the form $(\Lambda^{\text{cop}})^i_k(a) = \lambda^i_k(a_{(2)})a_{(1)}$ with $\lambda^i_k = \epsilon \circ (\Lambda^{\text{cop}})^i_k$.

Proof. (iii) is a B^{cop} version of (ii), (22) and (23). Taking into account (15) and (24), one calculates $a \cdot (e_k \otimes 1) = e_i \otimes \Lambda^i_k(a) = \lambda^i_k(a_{(1)})e_i \otimes a_{(2)} =$

$e_i \otimes \lambda^i_k(a_{(1)})a_{(2)}$. Hence $\Lambda^i_k(a) = \lambda^i_k(a_{(1)})a_{(2)}$. Applying ε to the both sides gives $\varepsilon \circ \Lambda^i_k(a) = \lambda^i_k(a_{(1)}\varepsilon(a_{(2)})) = \lambda^i_k(a)$. ■

Remark 4.5. The left–left $\mathscr{YD}$ condition (8) can be now rewritten as

$$\Lambda^i_m(a)L^m_k = L^i_m(\Lambda^{\mathrm{cop}})^m_k(a)$$

A similar statement holds true for left (free) covariant bimodules. A B-bimodule which is at the same time a B-bicomodule satisfying left and right Hopf B-module conditions together with the left and right covariance condition is called a Hopf B-bimodule or, in the terminology of Woronowicz, a *bicovariant* bimodule.

Assume that $M \doteq {}_BV$ is a left free bicovariant bimodule. In this case, apart from the right multiplication (20), one has a right comultiplication

$$_M\triangle\colon\quad B \otimes V \to B \otimes V \otimes B \quad \text{such that} \quad {}_M\triangle(a \cdot x \cdot b) = \triangle(a)_M\triangle(x)\triangle(b) \tag{25}$$

and the bicomodule property

$$(\mathrm{id} \otimes {}_M\triangle) \circ \triangle_M = (\triangle_M \otimes \mathrm{id}) \circ {}_M\triangle \tag{26}$$

Here, $\triangle_M(a \otimes v) = a_{(1)} \otimes a_{(2)} \otimes v$ denotes a free left comultiplication in M. In this case, the right comultiplication ${}_M\triangle$ in M is induced from a right comultiplication ${}_V\triangle$ in V,

$$_M\triangle(1 \otimes v) = 1 \otimes {}_V\triangle(v) \tag{27}$$

The structure theorem (Drinfeld, 1986; Woronowicz, 1989; Yetter, 1991) asserts that the vector space V equipped with the right multiplication (21) and the right comultiplication (27) inherits a right crossed B-module structure. The inverse statement is also true: a left (or right) crossed B-module structure on V generates a left (right) free Hopf B-bimodule structure on V_B (${}_BV$).

Remark 4.6. Due to Sweedler's (1969) theorem, any bicovariant bimodule M over a Hopf algebra H is free, i.e., it can be represented as ${}_HU$ or W_H, where $U \equiv (U, {}_Um, {}_U\triangle) \in \mathscr{SD}^H_H$ (resp., $W \equiv (W, m_W, \triangle_W) \in {}^H_H\mathscr{YD}$) denotes a crossed bimodule of left (resp. right) invariant elements in M, Remark 4.3. If the antipode S of H is bijective, then the following holds: $U \cong W$, ${}_Wm = (S^{-1})^*(m_U)$ and ${}_W\triangle = S_*({}_U\triangle)$, Proposition 3.3 (Radford and Towber, 1993).

5. BIMODULES DUAL TO FREE HOPF MODULES

For an arbitrary left A-module M, one can introduce its A-dual, a right A-module ${}^\dagger M \doteq \mathrm{Mod}\,(M, A)$, as a collection of all left module maps from

M into A (Bourbaki, 1989). The evaluation map gives a canonical A-bilinear pairing

$$\forall x \in M \quad \text{and} \quad X \in {}^{\dagger}M, \qquad \langle\langle a \cdot x, X \cdot b\rangle\rangle \doteq a \cdot X(x) \cdot b \in A \tag{28}$$

Similarly, for a right A-module N, $N^{\dagger} \doteq \mathrm{Mod}(N, A)$, a collection of all right module maps, seen as a left A-module, is called an A-dual of N. In this case, we shall write a canonical pairing

$$\forall y \in N \quad \text{and} \quad Y \in N^{\dagger}, \qquad \langle\langle a \cdot Y, y \cdot b\rangle\rangle \doteq a \cdot Y(y) \cdot b \in A \tag{29}$$

For free, finitely generated modules, one can repeat the dual basis construction.

Lemma 5.1 (Bourbaki, 1989). Let V be a finite-dimensional $\Bbbk$-space. A dual module to the left (right) free module ${}_AV$ (V_A) can be represented as $\tilde{V}_A$ (${}_A\tilde{V}$),

$${}^{\dagger}({}_AV) = \tilde{V}_A \quad \text{and} \quad (V_A)^{\dagger} = {}_A\tilde{V}, \qquad ({}^{\dagger}({}_AV))^{\dagger} = {}_AV \quad \text{and} \quad {}^{\dagger}((V_A)^{\dagger}) = V_A$$

The canonical A-bilinear pairing $\langle\langle\ ,\ \rangle\rangle$: ${}_A\tilde{V} \otimes V_A \to A$ can be rewritten by means of the $\Bbbk$-bilinear pairing $\langle\ ,\ \rangle$: $\tilde{V} \otimes V \to \Bbbk$,

$$a \otimes \alpha \in {}_A\tilde{V} \quad \text{and} \quad v \otimes b \in V_A, \tag{30}$$
$$\langle\langle a \otimes \alpha, v \otimes b\rangle\rangle = ab\langle\alpha, v\rangle \doteq ab\alpha(v)$$

Assume now that B is a bialgebra and we have done a right coaction ${}_V\triangle$: $V \to V \otimes B$. The image of ${}_V\triangle$ belongs to a right free B-module V_B. On the other hand, the transpose left action $\widetilde{{}_V\triangle}$: $\tilde{V} \to {}_B\tilde{V}$ takes its values in ${}_B\tilde{V}$, B-dual to V_B. This suggest a possibility for comparison between both pairings:

Lemma 5.2. Let B be a Hopf algebra with antipode S and a finite-dimensional space V is a right B-comodule with a coaction ${}_V\triangle$. Then

$$\forall v \in V \quad \text{and} \quad \alpha \in \tilde{V}, \qquad 1_B\langle\alpha, v\rangle = \langle\langle \widetilde{{}_V\triangle}(\alpha), S_*({}_V\triangle^{\mathrm{cop}})(v)\rangle\rangle \tag{31}$$

Proof. It is enough to check (31) on basis vectors due to (4), (7), (30) and using the properties of the antipode,

$$\langle\langle \widetilde{{}_V\triangle}(e^k), S_*({}_V\triangle^{\mathrm{cop}})(e_i)\rangle\rangle = \langle\langle R^k_j \otimes e^j, e_m \otimes S(R^m_i)\rangle\rangle = R^k_j S(R^j_i) = 1_B\delta^k_i \quad \blacksquare$$

If M is an A-bimodule, then ${}^{\dagger}M \doteq \mathrm{Hom}_{(A,—)}(M, A)$ can be equipped in a canonical bimodule structure (Borowiec, 1996) by

$$\langle\langle x, a \cdot X \cdot b\rangle\rangle \doteq X(x \cdot a) \cdot b = \langle\langle x \cdot a, X\rangle\rangle \cdot b \tag{32}$$

We call the bimodule ${}^{\dagger}M$ a left A-dual of M. Similarly, one can define a right dual $M^{\dagger} \doteq \mathrm{Hom}_{(—,A)}(M, A)$.

Let V_A be a right free bimodule with a left module structure given by the commutation rule (15), Then its right A-dual ${}_A\tilde{V}$ is a left free bimodule with the transpose commutation rule,

$$\begin{aligned}\langle\langle 1 \otimes e^k, a \cdot (e_i \otimes 1)\rangle\rangle &= \langle\langle 1 \otimes e^k, e_m \otimes \Lambda_i^m(a)\rangle\rangle \\ &= \langle e^k, e_m\rangle \Lambda_i^m(a) = \Lambda_i^k(a) \\ \langle\langle 1 \otimes e^k, a \cdot (e_i \otimes 1)\rangle\rangle &= \langle\langle (1 \otimes e^k) \cdot a, e_i \otimes 1\rangle\rangle \\ &= \langle\langle \Phi_m^k(a) \otimes e^m, e_i \otimes 1\rangle\rangle = \Phi_i^k(a)\end{aligned}$$

Therefore, $\Phi = \tilde{\Lambda}$. Assume further that A is a bialgebra, hence Λ has the form (24)

$$\Lambda(a) = \lambda(a_{(1)})(a_{(2)}) \tag{33}$$

for some representation λ of A in V. This means that V_A is a right A-covariant Hopf bimodule generated by (V, λ). Thus the transpose commutation rule $\Phi(a) = \tilde{\lambda}(a_{(1)})a_{(2)}$ makes ${}_A\tilde{V}$ a left A^{cop}-covariant Hopf bimodule (23).

These prove our main result:

Main Theorem 5.3. Assume V is a finite-dimensional vector space and A is an algebra.

(i) Let V_A be a right free A-bimodule whose left module structure is given by a commutation rule Λ: $A \to A \otimes \mathrm{End}\, V$. Then its right A-dual $(V_A)^\dagger = {}_A\tilde{V}$ is a left free A-bimodule with the transpose (right) commutation rule $\Phi = \tilde{\Lambda}$.

(ii) Assume further that A is a bialgebra and V_A is a right A-covariant bimodue generated by the representation λ: $A \to \mathrm{End}\, V$, (33). Then its right A-dual ${}_A\tilde{V}$ is a left (free) A^{cop}-covariant bimodule generated by $(\tilde{V}, \tilde{\lambda})$.

Theorem 5.3 suggests that dualizing a bicovariant bimodule over a bialgebra B, one obtains, in general, a B^{cop}-covariant bimodule which is not necessarily bicovariant unless $B \cong B^{\mathrm{cop}}$. However, in the case of the quantum group we get the following result.

Corollary 5.4. Let B be a bialgebra with a bijective antipode (quantum group). Let ${}_BV$ be a bicovariant B-bimodule generated by a right–right $\mathscr{YD}$ module $(V, {}_Vm, {}_V\Delta) \in \mathscr{YD}_B^B$. Then its left B-dual $\tilde{V}_B$ is a right (free) B^{cop}-covariant bimodule generated by $(\tilde{V}, \widetilde{{}_Vm})$. Due to Corollary 3.4 and Proposition 3.5, it can be also equipped with a bicovariant B^{cop}-bimodule structure generated by $(\tilde{V}, \widetilde{{}_Vm}, (S^{-1})_* (\widetilde{{}_V\Delta}^{\mathrm{cop}}) \in {}^{B^{\mathrm{cop}}}_{B^{\mathrm{cop}}}\mathscr{YD}$. Thus the identity (31) is satisfied.

Corollary 5.4 corrects the claim by Aschieri and Schupp (1996) and by Aschieri (1999) that general vector fields for a bicovariant differential calculus form a bicovariant bimodule over the same Hopf algebra.

Corollary 5.4 can be relevant for adapting the vector field formalism (Borowiec, 1996, 1997) to the case of differential calculus on quantum groups (Woronowicz, 1989).

Consider a right-covariant differential calculus $d: B \to \Gamma$ over a bialgebra B with values in a free right covariant B-bimodule Γ of one-forms. Thus $\Gamma \cong V_B$ and the left B-module of right invariant elements $V \equiv (V, \lambda)$ generates the left B-module structure of Γ, (24). A derivation $d: B \to \Gamma$ is said to be right-covariant if d is a right comodule map (Woronowicz, 1989, pp. 129–131, conditions 4 in Propositions 1.2–1.3);

$$_{\Gamma}\triangle \circ d = (d \otimes \mathrm{id}) \circ \triangle \tag{34}$$

Oziewicz (1998) considered an extension of a derivation, i.e., a short exact sequence of the differential algebras (algebras with derivations), in a similar spirit to Eilenberg's (1948) introduction of a bimodule over general algebra in terms of an extension of an algebra. Extension of a derivation and/or coderivation gives the seven Leibniz conditions and it appears that the Woronowicz conditions of the bicovariant derivation d, like (34), are among Leibniz conditions (Oziewicz, 1998, p. 216, formula (5.2)). In particular, (34) follows if one consider an extension of zero coderivation of B.

The right dual $\Gamma^\dagger \cong {}_B\tilde{V}$, where the generating space $(\tilde{V}, \tilde{\lambda})$ is a right B-module, is a left B^{cop}-covariant bimodule. The generalized Cartan formula

$$X^\delta(f) \doteq \langle\langle X, df\rangle\rangle \tag{35}$$

allows us to associate with any element $X \in \Gamma^\dagger$ the corresponding $\Bbbk$-linear endomorphism $X^\delta \in \mathrm{End}_\Bbbk B$. This gives the action $\delta: \Gamma^\dagger \to \mathrm{End}_\Bbbk B$ which satisfies the axioms of a right Cartan pair (Borowiec, 1996, 1997).

Proposition 5.5. With the assumptions as above, for any element $\alpha \in \tilde{V}$, the corresponding endomorphism $(1 \otimes \alpha)^\delta: B \to B$ is a right comodule map,

$$\triangle \circ (1 \otimes \alpha)^\delta = ((1 \otimes \alpha)^\delta \otimes \mathrm{id}) \circ \triangle \tag{36}$$

Proof. Let $\{e_i\}$ be any basis in V and $\{e^i\}$ the dual basis in $\tilde{V}$. Define the generalized derivatives $\partial^i \in \mathrm{End}_\Bbbk B$ with respect to a given basis $\{e_i\}$ by

$$\partial^i \doteq (1 \otimes e^i)^\delta$$

Then $df = e_i \otimes \partial^i f$. Substituting this into equation (34) and comparing the coefficients in front of the same basic vectors gives rise to

$$\forall i, \qquad (\partial^i f_{(1)}) \otimes f_{(2)} = (\partial^i f)_{(1)} \otimes (\partial^i f)_{(2)}$$

which is equivalent to (36). ■

Let B be a Hopf $\Bbbk$-algebra with bijective antipode. Consider a Woronowicz bicovariant differential calculus $d: B \to \Gamma$, where Γ is B-bicovariant bimodule of one-forms. Thus $\Gamma \cong V_B$ and $(V, m_V, \Delta_V) \in {}^B_B\mathcal{YD}$ denotes a crossed bimodule of the right invariant elements of Γ, Remark 4.6. Dualizing the bimodule of one-forms, one obtains a $B^{\rm cop}$-bicovariant bimodule $\widetilde{(\Gamma)}^\dagger \cong {}_B\tilde{V}$ of vector fields, where now, by Corollary 3.4, $(\tilde{V}, \widetilde{\rm m}_V, (S^{-1})_*(\widetilde{\triangle^{\rm cop}_V}) \in \mathcal{YD}^{B^{\rm cop}}_{B^{\rm cop}}$ denotes the crossed module of the Woronowicz left invariant vector fields. The quantum Lie bracket structure on $\tilde{V}$ is induced by the Yang–Baxter operator (Woronowicz, 1989).

APPENDIX

We give here an alternative, i.e. by direct calculations, proof of Theorem 5.3 *ii)*. For this aim, it suffices to check that the right mutliplication (cf. (23–24))

$$(1 \otimes \alpha).a \doteq a_{(2)} \otimes \tilde{\lambda}(a_{(1)})a \tag{A1}$$

and the left comultiplication (see Remark 4.2): $\Delta_{A\tilde{V}}(b \otimes \alpha) \doteq b_{(2)} \otimes b_{(1)} \otimes \alpha$ in ${}_A\tilde{V}$ are related by the left A^{cop}-covariant condition (cf. (20))

$$\Delta_{A\tilde{V}}(1 \otimes \alpha).a) = \Delta_{A\tilde{V}}(1 \otimes \alpha)\Delta^{cop}(a)\ . \tag{A2}$$

Calculation of the left-hand side gives

$$\Delta_{A\tilde{V}}(a_{(2)} \otimes \tilde{\lambda}(a_{(1)})\alpha) = (a_{(2)})_{(2)} \otimes (a_{(2)})_{(1)} \otimes \tilde{\lambda}(a_{(1)})\alpha\ . \tag{A3}$$

From the other hand similar calculations for the right-hand side yield

$$a_{(2)} \otimes (1 \otimes \alpha).a_{(1)} = a_{(2)} \otimes (a_{(1)})_{(2)} \otimes \tilde{\lambda}((a_{(1)})_{(1)})\alpha\ . \tag{A4}$$

The proof is finished since

$$(a_{(2)})_{(2)} \otimes (a_{(2)})_{(1)} \otimes a_{(1)} = a_{(2)} \otimes (a_{(1)})_{(2)}) \otimes (a_{(1)})_{(1)} \tag{A5}$$

due to the coassociativity property; $(\Delta^{\rm cop} \otimes {\rm id}) \circ \Delta^{\rm cop} = ({\rm id} \otimes \Delta^{\rm cop}) \circ \Delta^{\rm cop}$.

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